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Transonic shear flow in a three-dimensional channel

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Inviscid transonic shear flow in a rectangular channel is considered; opposite walls are parallel except in the region of interest, where one pair of opposing walls form a nozzle-like constriction. The flow exhibits the essential features found in an axial-flow rotor of zero stagger angle, where the relative velocity is transonic, the constricted passage being similar to the channel formed between two adjacent blades. Analytical solutions, valid to second order, are presented for the case where the ratio of the order of the change in velocity caused by the variation in flow area to the order of the change in velocity across the channel due to the shear is unity. The case where this ratio is small compared with one is discussed, as is the problem formulation for a flow with a shock wave in the passage

1. Introduction

The problem considered is that of a three-dimensional channel, rectangular in cross-section and with a nozzle-like constriction formed by two walls, through which passes a transonic shear flow. The flow field exhibits the essential features found in an axial-flow rotor when the velocity relative to the blades is transonic, although other applications may exist. Thus the flow through the constriction is associated with that between the blades of a rotor, and the plane walls upstream and downstream of the constriction approximate the symmetry boundaries in the flow entering and leaving the rotor when the stagger angle is zero. The remaining two walls, parallel to each other, are associated with the hub and tip of the rotor. The linear gradient in the incoming flow models the gradient that occurs in the flow relative to the blades in a rotor as a result of the radial variation of the tangential velocity component. This is, then, an extension to three dimensions of the problem considered by Ackeret & Rott (1949). Numerical results for a given set of conditions have been given by Oliver & Sparis (1971).

In the present work asymptotic methods of solution are employed. Such methods have proved successful in dealing with transonic channel flows that are two-dimensional and unsteady (e.g. Adamson, Messiter & Liou 1978), and, as will be seen, provide essential simplifications in this problem, allowing analytical solutions to be obtained for two cases. The most striking feature of these solutions is the fact that even though the incoming flow has a spanwise gradient in the velocity, the first perturbation depends only on the co-ordinate in the flow direction; i.e. the problem reduces to that for a simple one-dimensional channel flow, to first order. Higher-order solutions then contain three-dimensional effects. These results depend on the fact that the incoming flow remains a uniform shear flow no matter what conditions may be imposed downstream of the flow constriction. Hence they are not valid for, say, a shear flow entering a stationary channel in the case where signals could propagate upstream and change the form of the profile of the incoming stream. However, they are applicable to the flow relative to a rotor, in which case the velocity vector is composed of the uniform flow component in the axial direction, the magnitude of which may be changed by signals propagating upstream, and a tangential component imposed upon the flow by the rotation of the rotor, which depends only upon the product of the radius and the rotational velocity. Thus a gradient, approximated here by a linear gradient, always exists in the incoming flow.

In the following, the flow is assumed to be inviscid, compressible, and transonic, and the gas is assumed to follow the perfect-gas law and to have constant specific heats. The incoming flow is taken to be uniform, with the exception that it has a velocity gradient in one direction and thus is rotational; this is different from the actual rotor problem, where the relative entering flow is irrotational. The analysis is only slightly complicated by this difference.

2. Formulation of the problem

The three-dimensional channel considered, the incoming shear flow, and the notation used are shown in figure 1. In the top view are shown the flow constriction, corresponding to the channel formed between two blades, and the parallel walls, corresponding to symmetry boundaries in the actual rotor problem. The case illustrated is that for which the channel is symmetrical, corresponding to a cascade with no stagger and with the blades aligned parallel to the incoming flow. The extension to the case where one nozzle wall (half-blade) is moved downstream relative to the other, to simulate stagger, is not difficult, but is more complex algebraically and will not be shown here. In the side view are shown the parallel walls corresponding to the hub ($z = 0$) and tip shroud ($z = b$), the incoming shear flow, and the leading and trailing edges of the flow constriction (blades). All lengths are made dimensionless with respect to the half-chord $\frac{1}{2}\bar{c}$ of the blades, and velocities with respect to the speed of sound $\bar{a}_0$ in the incoming flow; overbars indicate dimensional quantities. The pressure P and temperature T are dimensionless with respect to their values in the incoming flow, which is referred to by the subscript 0. The blade spacing and span are $2s$ and b respectively. The position of the sonic surface in the incoming flow is z_{s0}, and the values of the velocity components, temperature and pressure there are

$$U_0 = 1+\delta(z-z_{s0}), \quad V_0 = W_0 = 0, \qquad (1a, b, c)$$

$$T_0 = P_0 = 1, \qquad (1d, e)$$

where, for transonic flow, $\delta \ll 1$. In the top view, the wall shape is given by

$$y_w = \pm(s-\epsilon^2 f(x)) \quad (|x| \leqslant 1), \qquad (2a)$$

$$y_w = \pm s \quad (|x| > 1), \qquad (2b)$$

where the thickness of the constriction (blade) is of order $\epsilon^2 \ll 1$. The blade shape is assumed to remain unchanged across its span.

It has been shown (e.g. Adamson, Messiter & Richey 1974; Messiter & Adamson 1975) that in a two-dimensional channel flow, variations in wall shape of $O(\epsilon^2)$ cause variations in U that are $O(\epsilon)$; thus a typical expansion is $U = 1+\epsilon u_1 + \ldots$. Here, then, problems that may be considered are characterized by the order of δ/ϵ, where δ orders ΔU_0, the ΔU that occurs across the channel in the incoming flow (as in ($1a$)), and ϵ orders ΔU_c, say, the ΔU that occurs in the flow direction as a result of the nozzle-like constriction. For $\delta/\epsilon \ll 1$, then, $\Delta U_0 \ll \Delta U_c$, and the flow field is described in terms of small perturbations from a two-dimensional channel flow. For $\delta/\epsilon \gg 1$, on the other hand, $\Delta U_c \ll \Delta U_0$; the constriction is expected to add small changes to the incoming

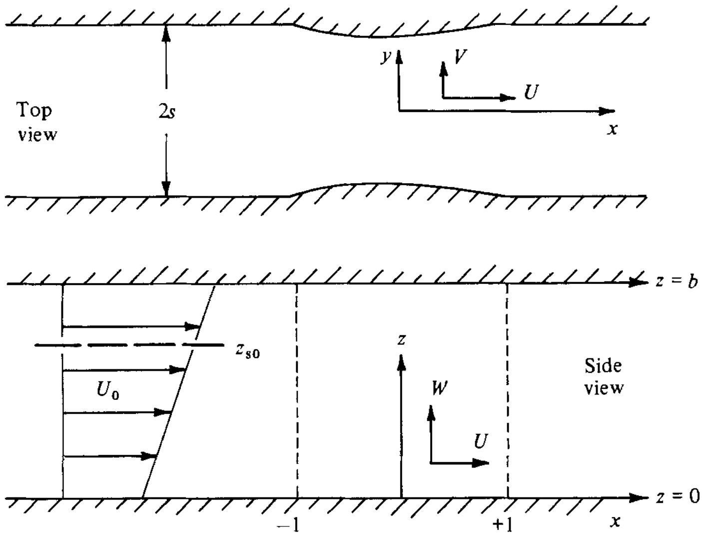

FIGURE 1. Top and side views of the rectangular channel, showing $U_0(z)$ and the position z_{s0} of the sonic surface in the incoming flow.

shear flow. Finally, for $\delta = O(\epsilon)$, the largest deviations from either two-dimensional or pure shear flow are to be expected.

The case $\delta = O(\epsilon)$ is considered here first, in detail, for conditions under which no shock waves occur in the channel. Then the problem formulation when a shock wave occurs in the supersonic part of the flow, still for $\delta = O(\epsilon)$, is presented; in this analysis it is found necessary to consider a thin inner region in which the solutions satisfy jump conditions where there is a shock wave and match with the outer channel-flow solutions upstream and downstream of the shock wave, in the appropriate limits. Finally, the case $\delta/\epsilon \gg 1$ is discussed briefly. When $\delta/\epsilon \ll 1$, solutions show relatively trivial differences from those for two-dimensional flow and so are not presented here. A very brief report of the initial phases of this work (Adamson 1977) shows only the first-order solutions for one velocity component, for $\delta = O(\epsilon)$.

3. Solutions for $\delta = O(\epsilon)$

For convenience, δ and ϵ are related as follows, where m is an arbitrary constant of order unity:

$$\delta = m\epsilon. \tag{3}$$

Solutions for the velocity components are written in terms of perturbations from their values in the incoming flow; because the flow has small rotation they are written in terms of a perturbation potential ϕ and an additional function $\mathbf{q}_A$ as follows:

$$\mathbf{q} = \mathbf{i}U_0 + \nabla\phi + \mathbf{q}_A. \tag{4}$$

Just as in problems in two-dimensional channel flow, the velocity components, and therefore ϕ, are written in terms of asymptotic expansions in powers of ϵ. Thus

$$\phi = \epsilon\phi_1 + \epsilon^2\phi_2 + \epsilon^3\phi_3 + \dots, \tag{5a}$$

$$U_A = \epsilon u_{a1} + \epsilon^2 u_{a2} + \epsilon^3 u_{a3} + \dots, \tag{5b}$$

with similar expansions for V_A and W_A. The location of the sonic surface in the incoming flow may be set to various orders of approximation also, and so is expanded as

$$z_{s0} = z_{00} + \epsilon z_{10} + \epsilon^2 z_{20} + \dots. \tag{6}$$

The stagnation enthalpy, made dimensionless with respect to $\bar{a}_0^2$, is

$$h_t = T/(\gamma - 1) + \tfrac{1}{2}(U^2 + V^2 + W^2) \tag{7}$$

and becomes $h_{t0} = 1/(\gamma-1) + \frac{1}{2}U_0^2 = h_{t0}\,(z)$ in the incoming flow, where γ is the ratio of specific heats. The expansion for h_t is then

$$h_t = h_{t0} + \alpha_1\, h_{t1} + \alpha_2\, h_{t2} + \dots, \tag{8}$$

where $\alpha_{i+1}(\epsilon) \ll \alpha_i(\epsilon) \ll 1$. Since $a^2 = T$, where a is the dimensionless (with $\bar{a}_0$) speed of sound, expansions for a or T may be found in terms of those already introduced by substituting the expansions for the velocity components, z_{s0} and h_t into (7). It may be noted that corresponding expressions for pressure and density can be derived, using the equations of state and the equation of motion in the flow direction.

The governing equations are, for steady inviscid flow,

$$\mathbf{q} \,.\, \nabla(\tfrac{1}{2}q^2) = a^2 \nabla \,.\, \mathbf{q}, \tag{9a}$$

$$\mathbf{q} \times \mathbf{\Omega} = \nabla h_t - \frac{T}{\gamma}\nabla S, \tag{9b}$$

$$(\mathbf{q} \,.\, \nabla)\, \mathbf{\Omega} = (\mathbf{\Omega} .\, \nabla)\, \mathbf{q} - \mathbf{\Omega}(\nabla \,.\, \mathbf{q}) + \frac{1}{\gamma}\nabla T \times \nabla S, \tag{9c}$$

where $\mathbf{\Omega} = \nabla \times \mathbf{q}$ is the vorticity and S is the entropy made dimensionless with respect to the gas constant $\bar{R}$. The incoming flow has uniform entropy and the flow is transonic, so that if shock waves occur, they are weak. In addition, because of the small curvature in the boundaries, the curvature of the shock wave is small; hence at a shock wave $U_{sh} = U_{sh}^{(0)}(z) + \dots$, and $S - S_0 = \epsilon^3 S_1(z) + \dots$ downstream of the shock wave. In general, (9c) is used to find $\mathbf{q}_A$, and Crocco's equation (9b) gives equations for the three derivatives of h_{t_i}; Crocco's equation is easier to use than the energy equation in this instance. Finally, the potential ϕ is found from the gasdynamic equation (9a).

The boundary conditions at the walls $z = 0$ and $z = b$ are simply

$$W(x, y, 0) = W(x, y, b) = 0, \tag{10}$$

while on the remaining walls

$$V(x, \pm y_w, z) = \mp \epsilon^2 f'(x)\, U(x, \pm y_w, z) \quad (|x| \leqslant 1), \tag{11a}$$

$$= 0 \qquad (|x| > 1), \tag{11b}$$

where $f'(x) = df/dx$. Finally, all perturbations die out as $x \to -\infty$, and at each level of approximation we choose, arbitrarily, to set the terms from $\nabla\phi$ and $\mathbf{q}_A$ equal to zero separately.

If the expansions for the velocity components, (4) with (1a), (3), (5) and (6), are substituted into (9c), it is found that $q_A = O(\epsilon^2)$; the flow is irrotational to first order. Using the same expansions and (8) for h_t, one finds from (9b) that $\alpha_1 = O(\epsilon^2)$; here we let

$$\alpha_1 = \epsilon^2, \tag{12}$$

where the arbitrary constant has been set equal to unity for convenience. Finally, if (7) with $T = a^2$ is substituted into (9a), use of the above mentioned expansions for the velocity components allows derivation of the first-order equation for ϕ; thus

$$\phi_{1yy}+\phi_{1zz} = 0, \tag{13}$$

where subscripts indicate partial differentiation. From (10) and (11) the corresponding boundary conditions are

$$\phi_{1y} = (x, \pm s, z) = 0, \tag{14a}$$

$$\phi_{1z}(x,y,0) = \phi_{1z}(z,y,b) = 0. \tag{14b}$$

Hence, from (13) and (14), it is seen that

$$\phi_1 = \phi_1(x). \tag{15}$$

That is, the first order perturbation is one-dimensional.

With (15) taken into account the equations for the second-order terms of $\mathbf{q}_\mathrm{A}$ found from the three components of ($9c$) are

$$[(w_{a2})_y-(v_{a2})_z]_x = 0, \tag{16a}$$

$$[(u_{a2})_z-(w_{a2})_x]_x = -m\phi_{1xx}, \tag{16b}$$

$$[(v_{a2})_x-(u_{a2})_y]_x = 0. \tag{16c}$$

Likewise, one finds from ($9b$) first that $(h_{t1})_x = 0$. Since all perturbations die out as $x \to -\infty$

$$h_{t1} = 0, \tag{17}$$

and so the remaining components of ($9b$) give equations that are integrals of ($16b$, c). If ($16a$–c) are integrated, the functions of integration are either identically zero or may be absorbed in ϕ_{2x}. Of the resulting partial differential equations only the particular solutions are required; ϕ_2 contains the corresponding complementary solutions. The particular solutions that satisfy the condition as $x \to -\infty$ are

$$u_{a2} = -mz\phi_{1x}, \quad v_{a2} = 0, \quad w_{a2} = 0. \tag{18a, b, c}$$

If (17) and (18) and the expansions for the velocity components and a^2 are employed in ($9a$), (10) and (11), one finds the following governing equation and boundary conditions for ϕ_2:

$$\phi_{2yy}+\phi_{2zz} = (\gamma+1)\,\phi_{1x}\,\phi_{1xx}+2u_0\,\phi_{1xx}, \tag{19a}$$

$$\phi_{2z}(x, y, 0) = \phi_{2z}(x, y, b) = 0, \tag{19b}$$

$$\phi_{2y}(x, \pm s, z) = \mp f'(x) \quad (|x| \leqslant 1), \tag{19c}$$

$$= 0 \qquad (|x| > 1), \tag{19d}$$

where

$$u_0 \equiv m(z-z_{00}). \tag{20}$$

The solution for ϕ_2 is found by substituting $\phi_2 = A(x, y)+B(x, z)$ into ($19a$) and letting $A_{yy} = (\gamma+1)\,\phi_{1x}\,\phi_{1xx}-g_0(x)$ and $B_{zz} = 2u_0\,\phi_{1xx}+g_0(x)$, where $g_0(x)$ is found through the application of the boundary conditions given in ($19b$). The result is

$$\phi_2 = [(\gamma+1)\,\phi_{1x}+m(b-2z_{00})]\,\tfrac{1}{2}y^2\,\phi_{1xx}+m\,(\tfrac{1}{3}z^3-\tfrac{1}{2}z^2b)\,\phi_{1xx}+h(x), \tag{21}$$

where $h(x)$ is a function of integration.

If ϕ_{2y} is calculated from (21), and then used in (19 *c*, *d*) in evaluating the boundary conditions, one obtains a governing equation for $\phi_1(x)$ which is easily integrated. Thus, one obtains

$$\phi_{1x} = -\frac{m}{\gamma+1}(b-2z_{00}) \qquad \left\{1 \mp \left[1-\frac{2(\gamma+1)f(x)}{sm^2(b-2z_{00})^2}\right]^{\frac{1}{2}}\right\}, \tag{22}$$

where the upper (minus) sign is used for $x < x_m$ and, when a shock wave occurs, for $x > x_{sh}$; the lower (positive) sign is used for $x_m < x < x_{sh}$. Here x_m is the position of the minimum in the cross-sectional flow area and x_{sh} refers to the position of a shock wave that might occur downstream of x_m; the conditions for the occurrence of such a shock wave will be discussed later. The condition that $\phi_{1x} \to 0$ as $x \to -\infty$ was used to set a constant in (22). With the use of (22), (21) may be simplified to

$$\phi_2 = -\frac{f'}{2s}y^2 + m(\tfrac{1}{3}z^3 - \tfrac{1}{2}z^2 b)\,\phi_{1xx} + h(x). \tag{23}$$

In (22), the sign of ϕ_{1x} depends upon the sign of $\frac{1}{2}b - z_{00}$; i.e. for $z_{00} \gtrless \frac{1}{2}b$, $\phi_{1x} \gtrless 0$, for $x \leqslant x_m$. Now, if $z_{00} > b$ the flow entering the channel is completely subsonic (the position of the sonic surface, for the given velocity gradient, lies outside the channel) and so one expects the flow to accelerate, $\phi_{1x} > 0$, as it moves through the part of the flow constriction where the cross-sectional area is decreasing. Similarly, one expects decelerated flow, $\phi_{1x} < 0$, for $z_{00} < 0$ where the entering flow is entirely supersonic. Here, however, the result is that if the flow is mixed, but subsonic on the average ($z_{00} > \frac{1}{2}b$) or mixed and supersonic on the average ($z_{00} < \frac{1}{2}b$), one finds $\phi_{1x} > 0$ or $\phi_{1x} < 0$ respectively.

The equation for the sonic surface, $z_s = z_0 + \epsilon z_1 + \ldots$, is found by setting $q = a$ and using equation (7) for $a = T^{\frac{1}{2}}$. To lowest order, the result is

$$z_0 = \tfrac{1}{2}b \mp (\tfrac{1}{2}b - z_{00})\left[1-\frac{2(\gamma+1)f(x)}{sm^2\,(b-2z_{00})^2}\right]^{\frac{1}{2}}, \tag{24}$$

where the sign conventions mentioned for (22) hold for (24) as well.

In both (22) and (24) it is seen that if $f(x_m) = t$ is the point at which $f(x)$ is a maximum so that the cross-sectional area is a minimum, then for real values of ϕ_{1x} and z_0 the condition

$$|z_{00} - \tfrac{1}{2}b| \geqslant \left[\frac{(\gamma+1)\,t}{2sm^2}\right]^{\frac{1}{2}} \tag{25}$$

must be met. Thus there is a forbidden region for the sonic surface in the incoming flow. This region is illustrated by calculations shown in figure 2, as are solutions for the sonic surface for various values of z_{00}, for a circular-arc airfoil. For z_{00} greater or less than the limits found from (25) using the equality (i.e. outside the forbidden region), only the upper sign in (24) may be used; for z_{00} equal to either limit, the upper sign is used for $x < x_m$ and either the upper or lower sign is used for $x > x_m$. No shock waves are considered. It is seen that since the flow velocity consists of a linear gradient plus a perturbation that depends only upon x, to first order, the velocity retains the linear gradient throughout the channel; the average velocity at any x-position is its value at $z = \frac{1}{2}b$. Hence in figure 2 it is seen that the boundaries of the so-called forbidden region correspond to those average values of the incoming velocity (one subsonic and one supersonic) for which the average value of the velocity at the minimum area is sonic (the sonic surface passes through $z = \frac{1}{2}b$ at $x = x_m$). This is,

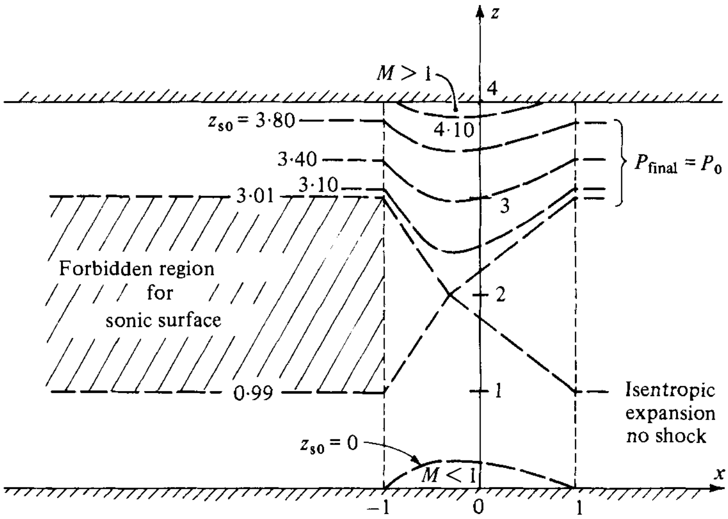

FIGURE 2. Sonic surface (first order) z_0 versus x for various initial values, z_{s0}, from (24). Airfoil has circular-arc profile, $f(x) = t(1-(x-x_m)^2/(1 \pm x_m)^2)$, with upper sign for $x < x_m$; $b = 4$, $t = 2$, $\epsilon = 0{\cdot}1$, $s = 6$, $m = 0{\cdot}625$, $x_m = -\frac{1}{3}$. (Aspect ratio = 2 and airfoil thickness ratio = 2 %.)

then, on the average, precisely the same situation found in two-dimensional nozzle flows as a choking condition is approached with either subsonic or supersonic values for the velocity upstream of the throat of the nozzle.

Again referring to figure 2, it is clear that for each z_{00} greater than the upper limit there is a corresponding pressure downstream of the constriction. As the pressure downstream decreases, signals can pass upstream and change the conditions in the incoming flow such that $z_s = z_{00} + \ldots$ decreases, but the gradient remains. This corresponds, in a compressor flow, to signals moving upstream and increasing the value of the absolute axial flow with the tangential component of velocity (rw) remaining unchanged. This result holds until the upper limit predicted by (25) is reached, at which time the average velocity at the minimum area is sonic. As shown on figure 2, there are also two downstream conditions for which this averaged velocity-choking condition holds. Evidently, for pressures between those associated with these two conditions, there is a shock wave in the channel, downstream of x_m, which does not extend across the channel and which moves downstream as the pressure downstream of the constriction decreases. This explanation contains the implicit assumption that the flow at some point upstream of the shock wave is unchanged, even though conditions downstream of the shock wave are varied. It will be shown that one can in fact consider an inner region about the shock wave, $O(\epsilon^{\frac{1}{2}})$ in thickness, in which signals can pass upstream of the shock wave in the subsonic part of the flow and thus affect the supersonic flow upstream of the shock. However, this effect is evidently a local one; the subsonic and supersonic parts of the flow accommodate to each other in a thin region, and one can continue to explain variations in the flow by applying simple one-dimensional flow concepts to the average flow quantities. Evidently, this mixed channel flow can be choked.

In order to complete the solutions to second order, it is necessary to find h_x in (23); as in the analysis for ϕ_{1x}, this involves finding the third-order terms in the V and W velocity components and applying the boundary conditions. Following the same sequence of steps used in obtaining the governing equations for ϕ_{1x}, one finds for h_x

$$\begin{aligned} h_x = & -\tfrac{1}{6}(2\gamma-3)\,\phi_{1x}^2+\tfrac{1}{6}f''s+\tfrac{1}{12}mb^3\,\phi_{1xxx} \\ & +[(\gamma+1)\phi_{1x}+m(b-2z_{00})]^{-1}\left\{\frac{mz_{00}f}{s}-\tfrac{1}{6}mbf''s\right. \\ & +\tfrac{1}{2}\phi_{1}^2x\left[-m(b-2z_{00})\tfrac{1}{6}(5\gamma-3)+mb(\gamma+1)\right]+2m^2b^4\,\phi_{1xxx} \\ & \left.+m^2(\tfrac{1}{2}b^2-z_{00}^2)\,\phi_{1x}+C_2\right\}+\frac{2mz_{10}}{\gamma+1}, \end{aligned} \tag{26}$$

where C_2 is a constant of integration. It is seen that when the flow is choked to first order and $x \to x_{\mathrm{m}}$, then $(\gamma+1)\,\phi_{1x}+m(b-2z_{00}) \to 0$, and h_x may become singular. However, using (22), one can show that if near $x = x_{\mathrm{m}}$, $f(x)$ has the typical form $1-f(x)/t = \text{const}\ (x-x_{\mathrm{m}})^2 + \dots$, then ϕ_{1xxx} does not exhibit singular behaviour. Under these conditions, if h_x is to remain finite,

$$C_2 = \tfrac{1}{6}mbf''_m-2m^2b^4(\phi_{1xxx})_m+\frac{mt}{s(b-2z_{00})}\left\{bz_{00}+\tfrac{1}{6}(b-2z_{00})^2\frac{5\gamma-3}{\gamma+1}\right\}. \tag{27}$$

Moreover, for $U = U_0$ as $x \to -\infty$, then

$$z_{10} = -\frac{(\gamma+1)\,C_2}{2m^2(b-2z_{00})}, \tag{28}$$

and solutions for the velocity components, valid to $O(\epsilon^2)$, have been obtained. The shape of the stream surfaces are easily derived from these solutions.

4. Problem formulation when a shock wave occurs ($\delta = O(\epsilon)$)

The conditions under which a shock wave may be expected to occur are those for which the pressure immediately downstream of the flow constriction is between the values associated with the limiting position of the sonic surface. In figure 2, it is seen that since flow conditions are multi-valued only for $x > x_{\mathrm{m}}$, then $x_{\mathrm{sh}} > x_{\mathrm{m}}$, where, again, x_{sh} refers to the position of the shock wave. Moreover, since the pressure downstream of the constriction must fall between the limiting values, and the pressure across a shock wave increases, the flow upstream and downstream of the shock wave must be, on the average, supersonic and subsonic respectively; in (22) and (24), then, the upper signs hold for $x_{\mathrm{m}} < x < x_{\mathrm{sh}}$ and the lower for $x > x_{\mathrm{sh}}$. Now, it can be shown that the solutions for U do not satisfy the jump conditions across a shock wave even to $O(\epsilon)$, so that an inner region enclosing the shock wave must be considered, as in the case of a flow in a two-dimensional channel (Messiter & Adamson 1975). The solutions given so far may therefore be considered as outer solutions to which the solutions in this inner region must be matched; in addition, the jump conditions across the shock wave must be satisfied by the inner solutions.

Although there are some similarities to the problem of a shock wave in the flow through a two-dimensional channel, there is a fundamental difference in that in the present case the shock wave need not extend across the channel. An idea of the flowfield that results in this case may be gained by noting that at the upper surface the shock wave must be normal to the wall, and that the sonic surface must join the curve given by (24) as the outer region is approached. A sketch of the resulting configuration for accelerating flow is shown in figure 3, where the details of the

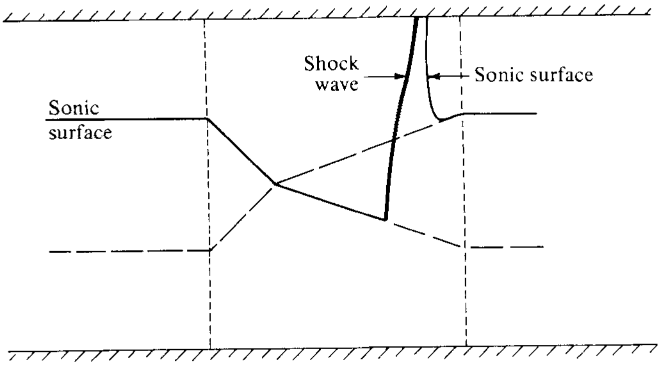

FIGURE 3. Sketch of flow field with a shock wave. Details of the region where the shock wave meets the sonic surface, omitted here, are shown in figure 4. Dashed lines show limiting sonic surfaces (for flows in which the average Mach numbers are unity at the location of minimum cross-sectional flow area). In the case shown, the sonic surface follows the limit shape for accelerating flow up to the shock wave. Downstream of the shock, it joins the limit shape for flow that is subsonic (on the average) to first order.

formation of the shock wave near the sonic surface are not shown; evidently, this formation occurs as a result of the coalescence of weak compression waves emanating from the sonic surface.

In the inner region there must be a rapid adjustment of the flow from upstream to downstream conditions. Therefore, it is to be expected that flow acceleration will be important to first order. Variations in U are $O(\epsilon)$, and if in this inner region $x-x_{\mathrm{sh}} = O(\epsilon^{\alpha})$, say, then a perturbation potential would be $O(\epsilon^{1+\alpha})$, and y and z are both $O(1)$. From (9a), then, it is easily shown that $\alpha = \frac{1}{2}$. Hence, in the inner adjustment region, the velocity components are taken to be functions of x^+, y and z, where x^+ and the expansions for these components are as follows:

$$x^+ = (x-x_{\mathrm{sh}})\,\epsilon^{-\frac{1}{2}}, \tag{29a}$$

$$U = 1+\epsilon u_1^+ + \epsilon^{\frac{3}{2}} u_2^+ + \dots, \tag{29b}$$

$$W = \epsilon^{\frac{3}{2}} w_1^+ + \epsilon^2 w_2^+ + \dots, \tag{29c}$$

with an expansion for V similar to that for W. In (29b), $u_i^+ = \phi_{ix^+}^+ + u_{ia}^+$; i.e. the velocity is again a sum of a potential and additional function, with similar relations for the v_i^+ and w_i^+.

The solutions to which those in the inner adjustment region must match as $x^+ \to \pm\infty$ are found by expanding the outer solutions derived in §3 about $x = x_{\mathrm{sh}}$. To lowest order, they are

$$U = 1+\epsilon m\left\{z-\left[z_{00}+\frac{b-2z_{00}}{\gamma+1}(1\pm g_{\mathrm{sh}}^{\frac{1}{2}})\right]\right\}+\dots, \tag{30a}$$

$$V = -\epsilon^2\frac{y}{S}f'(x_{\mathrm{sh}})+\dots, \tag{30b}$$

$$W = \mp\epsilon^2\frac{m^2}{\gamma+1}(z^2-bz)(b-2z_{00})\frac{g'_{\mathrm{sh}}}{g_{\mathrm{sh}}^{\frac{1}{2}}}+\dots, \tag{30c}$$

where g_{sh} represents the radicand in the radical in (24), evaluated at $x = x_{\mathrm{sh}}$, and where the upper and lower signs hold as $x^+ \to -\infty$ and $x^+ \to +\infty$ respectively. The boundary conditions are simply

$$v_{\mathrm{I}}^+(x^+, \pm s, z) = 0, \tag{31a}$$

$$w_{\mathrm{I}}^+(x^+, y, 0) = w_{\mathrm{I}}^+(x^+, y, b) = 0. \tag{31b}$$

Using (9*a*–*c*) and the above expansions and matching and boundary conditions, it may be shown that $v_{\mathrm{I}}^+ = 0$, $u_{\mathrm{I}}^+ = u_{\mathrm{I}}^+(x^+, z)$ and $w_{\mathrm{I}}^+ = w_{\mathrm{I}}^+(x^+, z)$ satisfy all conditions and equations, and that the governing equations for u_{I}^+ and w_{I}^+ are

$$[(\gamma+1)\,u_{\mathrm{I}}^+ - (\gamma-1)\,mz]\; u_{\mathrm{I}x^+}^+ - w_{\mathrm{I}z}^+ = 0, \tag{32a}$$

$$(u_{\mathrm{I}}^+ - mz)_z - w_{\mathrm{I}x^+}^+ = 0. \tag{32b}$$

Finally, from the equations for the shock polar and the shape of the shock wave (Messiter & Adamson 1975), it is found that at the shock,

$$[\tfrac{1}{2}(\gamma+1)\,(u_{\mathrm{Id}}^+ + u_{\mathrm{Iu}}^+) - (\gamma-1)\,mz]\,(u_{\mathrm{Id}}^+ - u_{\mathrm{Iu}}^+)^2 = (w_{\mathrm{Id}}^+ - w_{\mathrm{Iu}}^+)^2, \tag{33a}$$

$$\frac{dx_{\mathrm{s}}^+}{dz} = -\frac{w_{\mathrm{Id}}^+ - w_{\mathrm{Iu}}^+}{u_{\mathrm{Id}}^+ - u_{\mathrm{Iu}}^+}, \tag{33b}$$

where subscripts u and d refer respectively to conditions immediately upstream and downstream of the shock wave. The problem, then, is the solution of (32*a*, *b*), subject to the given boundary (31) and matching (30) conditions, with (33*a*, *b*) holding at the shock wave in the interior of the region. At the upper wall, where the shock wave is normal, a jump in pressure occurs. As z decreases, the shock wave becomes weaker; thus the pressure jump across the shock becomes smaller with a larger change in pressure upstream of the wave due to the many compression wavelets that arise from the sonic surface and coalesce with the shock wave, making it stronger with increasing z. Finally, the strength of the shock goes to zero as the sonic surface is approached. In the subsonic region, the pressure gradient is supported by a relatively large curvature of the streamlines. A sketch of the flow problem to be solved, to the scale of the inner region, is shown in figure 4.

It is clear that a numerical solution is called for in this inner adjustment region. Except for the boundary conditions along the upper wall, the problem is quite similar to that considered by Melnick & Grossman (1974) in their study of the interaction between a shock wave and a boundary layer. Because this is a model problem, it was not deemed worthwhile to expend the considerable effort necessary to perform numerical computations. However, a very simple three-strip method of integral-relations solution was used to obtain some numerical results. Although the accuracy obtained was not sufficient to warrant presentation, the general features described above and the similarities to the work of Melnik & Grossman (1974) were confirmed.

It should be noted that in the outer solutions, to which the solutions in this inner region must be matched, the term of $O(\epsilon)$ in the pressure (the first perturbation) does not have a gradient in z, in contrast to the velocity; it depends only upon ϕ_{1x} and is thus a function of *x alone*. This can be seen immediately upon consideration of the equation of motion in the flow direction. Evidently, this allows the formulation of the inner region as described above; no distinction need be made, as $x^+ \to -\infty$, between that part of the flow for which $z > z_{\mathrm{s}}$ and that for which $z < z_{\mathrm{s}}$. That is, there need be no distinction, to first order, between the subsonic and supersonic parts of the flow. Therefore, although the supersonic flow upstream of the shock wave is dependent upon conditions downstream of it because the flow is mixed in the

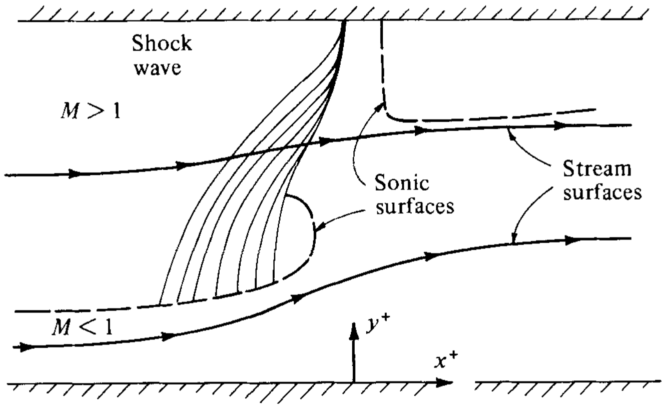

FIGURE 4. Sketch of the flow field to scale of inner region enclosing the shock wave ($x = O(\epsilon^{\frac{1}{2}})$) showing the compression wavelets that coalesce to form the shock wave, as in Melnik & Grossman (1974).

z-direction, to $O(\epsilon)$ this upstream influence is limited to a distance $O(\epsilon^{\frac{1}{2}})$ upstream of the shock. Thus the essential result of changes in pressure downstream of the shock wave will be changes in its location; by analogy with the result for flow in two-dimensional channels (Messiter & Adamson 1975) it is expected that variations in back pressure $O(\epsilon^2)$ will cause changes in location of the shock wave $O(1)$. Although the problem under consideration is only a model of the actual flow through a rotor, so that detailed comparisons are not valid, it appears that the general results found here should apply.

5. Solutions for $\delta \gg \epsilon$

As mentioned previously, when $\delta \gg \epsilon$ one expects small changes in the flow as it passes through the constriction. Here, in a brief analysis, it is shown that as long as $z_{00} - \frac{1}{2}b = O(1)$, the perturbations to the flow are indeed of higher order than in the $\delta = O(\epsilon)$ case. However, as the sonic surface in the incoming flow approaches the centreline of the channel, the possibility of choking again occurs and larger perturbations are found. In the following, $\delta = O(\epsilon^{\frac{1}{2}})$ is chosen as an example of the general condition $\delta \gg \epsilon$.

When $\delta = \epsilon^{\frac{1}{2}}$, multiple half-powers of ϵ are necessary in the expansions for the velocity components. Thus it is found that the correct expansions are

$$U = 1 + \epsilon^{\frac{1}{2}} m (z - z_{S0}) + \epsilon^{\frac{3}{2}} \phi_{\frac{3}{2}x} + \epsilon^2 u_2 + \dots, \tag{34a}$$

$$V = \epsilon^2 v_2 + \dots, \tag{34b}$$

$$W = \epsilon^2 w_2 + \dots, \tag{34c}$$

where, following the same procedure as in the case where $\delta = O(\epsilon)$, one can show that the flow is irrotational to $O(\epsilon^{\frac{3}{2}})$ and that $\phi_{\frac{3}{2}} = \phi_{\frac{3}{2}}(x)$. Again, from (9*a*), the governing equation for second-order terms is

$$v_{2y} + w_{2z} = 2m(z - z_{00})\, \phi_{\frac{3}{2}xx}, \tag{35}$$

with boundary conditions

$$v_2(x, \pm s, z) = \mp f'(x), \tag{36a}$$

$$w_2(x, y, 0) = w_2(x, y, b) = 0. \tag{36b}$$

Division of (35) into two parts, $w_{2z} = 2m(z-z_{s0})\,\phi_{\frac{3}{2}xx} + G_0(x)$ and $v_{2y} = -G_0(x)$, allows derivation of solutions for w_2 and v_2. When the boundary conditions are satisfied a differential equation that is easily integrated is found for $\phi_{\frac{3}{2}}$. Finally, then, one obtains for U, V, and W,

$$U = 1+\epsilon^{\frac{1}{2}}m(z-z_{00})+\epsilon^{\frac{3}{2}}\frac{f(x)}{2ms\,(z_{00}-\frac{1}{2}b)}+\dots, \tag{37a}$$

$$V = -\epsilon^2 f'(x)\,y/s+\dots, \tag{37b}$$

$$W = -\epsilon^2\frac{b^2 f'(x)}{2s\,(z_{00}-\frac{1}{2}b)}\left[\frac{z}{b}-\left(\frac{z}{b}\right)^2\right]+\dots. \tag{37c}$$

From (37a–c), it is seen that, as long as $z_{00}-\frac{1}{2}b = O(1)$, perturbations in U caused by the constriction are $O(\epsilon^{\frac{3}{2}})$ and that V and W are $O(\epsilon^2)$, as in the case $\delta = O(\epsilon)$. Moreover, the solution for U is quite simple, with no limitations on the position of the sonic surface, which is, simply, $z_s = z_{00}-\epsilon(\gamma+1)\,\phi_{\frac{3}{2}x}+\dots$. No choking occurs.

As $z_{00}-\frac{1}{2}b$ becomes small, the solution is not so simple. For example, if (7), with $T = a^2$, and (37a) are used to calculate the Mach number M, it is easily shown that the average value (over z) of $M-1$ is, if $U = U_0+\epsilon\,\phi_{1x}+\dots$,

$$\overline{M-1} \equiv \frac{1}{b}\int_0^b [\epsilon^{\frac{1}{2}}m(z-z_{00})+\tfrac{1}{2}\epsilon\,(\gamma+1)\,\phi_{1x}+\dots]\,dz = \epsilon^{\frac{1}{2}}m(\tfrac{1}{2}b-z_{00})+O(\epsilon). \tag{38}$$

Thus, from (37a) and (38), it is seen that for $z_{00}-\frac{1}{2}b = O(\epsilon^{\frac{1}{2}})$ the first perturbation from U_0 is of the same order as $\overline{M-1}$; evidently this is a distinguished limit in which choking can occur, and must be investigated in more detail. An interesting feature of this case is that while $V = O(\epsilon^2)$, as in the previous calculations, W now is $O(\epsilon^{\frac{3}{2}})$, as seen from (37$c$).

If we now define z_{00}^* and write U_0 in terms of it as

$$z_{00}^* = (z_{00}-\tfrac{1}{2}b)\,\epsilon^{-\frac{1}{2}}, \tag{39a}$$

$$U_0 = 1+\epsilon^{\frac{1}{2}}m\,(z-\tfrac{1}{2}b)-\epsilon\,mz_{00}^*, \tag{39b}$$

then, the expansions for the velocity components may be written as

$$U = U_0+\epsilon\phi_{1x}+\epsilon^{\frac{3}{2}}u_{\frac{3}{2}}+\dots, \tag{40a}$$

$$V = \epsilon^2 v_2+\dots, \tag{40b}$$

$$W = \epsilon^{\frac{3}{2}}w_{\frac{3}{2}}+\epsilon^2 w_2+\dots. \tag{40c}$$

Following the same procedures as in the previous calculations, it is easily shown that the flow is irrotational to order ϵ, that there are no terms of $O(\epsilon)$ in V and W, and that $\phi_1 = \phi_1(x)$. From (9a) and (7), it is found that the equation for $w_{\frac{3}{2}}$ is

$$2m\,(z-\tfrac{1}{2}b)\,\phi_{1xx} = w_{\frac{3}{2}z}, \tag{41}$$

so that after integration and application of boundary conditions

$$w_{\frac{3}{2}} = mb^2\left[\frac{z^2}{b^2}-\frac{z}{b}\right]\phi_{1xx}. \tag{42}$$

This result agrees with that given in (37c) if the ϕ_{1x} shown in (37a) is used to calculate ϕ_{1xx}. In order to obtain ϕ_1 for the present limit, it is necessary to derive the equation involving w_2 and v_2, again from (9a) and (7). This is found to be

$$[(\gamma+1)\,\phi_{1x}-2mz_{00}^*+m^2(z-\tfrac{1}{2}b)^2]\,\phi_{1xx}+2m\,(z-\tfrac{1}{2}b)\,u_{\frac{3}{2}x}+mw_{\frac{3}{2}}-v_{2y}-w_{2z} = 0, \tag{43}$$

where it may be noted that since both ϕ_{1xx} and $u_{\frac{3}{2}x}$ appear, a second relation between them is necessary. This relation is derived by substituting (40*a–c*) into the vorticity equation (9*c*); one obtains $(w_{\frac{3}{2}})_{xy} = (u_{\frac{3}{2}})_{xy} = 0$ and $((u_{\frac{3}{2}})_z - (w_{\frac{3}{2}})_x)_x = -m\,\phi_{1xx}$. The first of these questions is satisfied by (42); the second and third indicate that $u_{\frac{3}{2}}$ is independent of y, and indeed $u_{\frac{3}{2}}$ may be found by integrating the third equation, after substitution of (42), to give

$$u_{\frac{3}{2}} = -m\phi_{1x}z + mb^2\left[\frac{z^3}{3b^2} - \frac{z^2}{2b}\right]\phi_{1xxx} + g(x), \tag{44}$$

where $g(x)$ is a function of integration.

If, again, (43) is divided such that

$$\begin{aligned} v_{2y} &= [(\gamma+1)\,\phi_{1x} - 2mz^*_{00}]\,\phi_{1xx} + g_0(x), \\ w_{2z} &= m^2(z-\tfrac{1}{2}b)^2\,\phi_{1xx} + 2m\,(z-\tfrac{1}{2}b)\,u_{\frac{3}{2}x} + mw_{\frac{3}{2}} - g_0(x), \end{aligned}$$

and (42) and (44) are used for $w_{\frac{3}{2}}$ and $u_{\frac{3}{2}}$ respectively in the equation for w_{2z}, then solutions that satisfy the boundary conditions may be found for w_2 and v_2. That is, $g_0(x)$ is found by application of the boundary conditions $w_2 = 0$ at $z = 0$ and $z = b$, and the governing equation for ϕ_1 is obtained when the boundary conditions $v_2(x, \pm s, z) = \mp f'$ are applied. Thus

$$\phi^2_{1x} - 2\left(\frac{2mz^*_{00}}{\gamma+1} + \frac{m^2b^2}{4(\gamma+1)}\right)\phi_{1x} = -\frac{2f}{s(\gamma+1)} - \frac{m^2b^4}{15(\gamma+1)}\phi_{1xxx} \tag{45}$$

defines $\phi_1(x)$ in terms of the wall shape $f(x)$. It may be noted, as a check, that for $m = O(\epsilon^{\frac{1}{2}})$ so that $\delta m = O(\epsilon)$, the result for ϕ_{1x} is again that given in (22). The appearance of ϕ_{1xxx}, the second derivative of the velocity perturbation, is of some interest because it has no counterpart in governing equations for flows in two-dimensional channels.

Although an analytical solution for $\phi_1(x)$ in terms of $f(x)$ cannot be found, it is possible to infer some key features of the flow field from (45). For this discussion, it is convenient to let $u_1 = \phi_{1x}$, $\alpha = 2mz^*_{00}/(\gamma+1)$, $\beta = m^2b^2/4(\gamma+1)$ and $g = 2f/s(\gamma+1) + m^2b^4u_{1xx}/15(\gamma+1)$. Then (45) and its derivative may be written as

$$u_1 = \alpha+\beta \pm \left[(\alpha+\beta)^2 - g\right]^{\frac{1}{2}}, \tag{46a}$$

$$u_1' = -\frac{g'}{2(u_1 - (\alpha+\beta))}, \tag{46b}$$

where the prime indicates a derivative with respect to x. These equations are similar to those relating Mach number and area in a one-dimensional channel flow. Here, the equivalent area function g is formed by the function f that governs the actual area, and an additional function u_{xx}, which has the same effect as f and thus may be interpreted as being equivalent to a change in cross-sectional flow area. Now, however, this function depends on the solution itself. This viewpoint is employed when heat addition to a nozzle flow is considered (Zierep 1975); although u_{xx} is certainly not an arbitrary function as in the flow with heat addition, there are, nevertheless, some similarities with this problem. In the present problem, the relatively large vorticity in the incoming flow leads, through continuity, to a relatively large value of w $(O(\epsilon^{\frac{3}{2}}))$. This, in turn, leads through the vorticity equation to increased acceleration of the fluid and a rate of change of acceleration that appears equivalent to an area change. It may be noted that if the cross-sectional flow area in the stream tube formed by the sidewalls, the wall at $z = 0$ (or $z = b$), and a chosen stream surface, $z = z_{ss}(x)$, is calculated, it is found that this area goes through an extremum when $w_{\frac{3}{2}} = 0$ and thus from (42) when $u_{1x} = 0$ rather than when $f' = 0$. This result lends some support to the present interpretation in that one possible cause for $g' = 0$ is $u_{1x} = 0$.

If the indicated integration in (38) is carried out to terms $O(\epsilon)$, it is easily shown that $\overline{M-1} \rightarrow 0$ as $u_1 \rightarrow \alpha$. Inspection of (46*b*), however, shows that the value of u, associated with $g' = 0$ and with the singularity when u_1 has a bifurcation (46*a*), is $u_1 = \alpha+\beta$. Apparently, then, for this case of large shear, the flow field does not reach the same conditions as a one-dimensional flow at a minimum area; the average Mach number at this point is no longer unity.

If one imagines an incoming flow that is subsonic on the average ($z^*_{00} > 0$, so $\alpha > 0$) and that accelerates to some maximum velocity, then it appears possible that as the maximum is approached, $u_{1xxx} < 0$ and although $u_{1xx} < 0$, $g > 0$ such that $g' = 0$ and $g > 0$ at the maximum where $u_{1x} = 0$. In this event, since $f' > 0$, $g' = 0$ upstream of the point at which $f' = 0$ (physical minimum area). In order that $u_1 = 0$ as $g \rightarrow 0$ upstream of the flow constriction, the minus sign is used in (46*a*). As the magnitude of the average Mach number in the incoming flow is increased, α decreases in magnitude. When conditions are such that as the fluid accelerates through the constriction, $g = (\alpha+\beta)^2$ at the point at which $g' = 0$, then $u_1 = \alpha+\beta$ there and $u_{1x} \neq 0$. Downstream of that point, then, either solution of (46*a*) is valid, depending upon downstream conditions. In fact, if one considers possible variations of u_{1xxx} and u_{1xx} throughout the flow, it appears possible that another point where $g' = 0$ may exist downstream of the first, similar to the situation that exists in the flow with heat addition (Zierep 1975). In any event, once the flow downstream of the point at which $g' = 0$ becomes supersonic, the possibility of shock waves in the supersonic part of the flow exists, as shown in the case $\delta = \epsilon$. However, in the present case, where the flow is governed by an effective area change involving u_{1xx}, rather than the physical-area change, questions of the stability of the flow with a shock wave would have to be investigated.

Finally, the counterpart of the above example, that where the incoming flow is supersonic on the average ($\alpha < 0$), can be analysed in the same manner by imagining a flow that decelerates to a minimum value of u_1, with $u_{1xxx} > 0$ and $u_{1xx} > 0$ so that $g > 0$ as $u_{1x} = 0$ is approached. In this case, $f' < 0$ if $g' = 0$ is to occur, so the extremum in g occurs downstream of the point $f' = 0$. The condition $\alpha < 0$ such that $\alpha+\beta < 0$ is considered because if $\alpha < 0$ such that $\alpha+\beta > 0$, then from (46*b*) $u_1 > \alpha+\beta$ if u_1' is to be negative (decelerating flow), and the positive sign must be used in (46*a*); then, however, the condition that $u_1 \rightarrow 0$ upstream of the constriction where $g \rightarrow 0$ cannot be met. For $\alpha+\beta < 0$, the positive sign is used, and, as in the previous example, a singularity occurs if $u_1 = \alpha+\beta$ at the point where $g' = 0$; two possible solutions exist downstream of the flow constriction. Again, shock waves in the supersonic part of the flow may be possible, but questions of shock-wave stability must be answered before this can be considered to be a certainty.

6. Discussion

A general conclusion drawn from this study and which may be applied to the flow through a three-dimensional compressor cascade is that to lowest order the flow may be interpreted the same way as a one-dimensional flow so long as the average Mach number is considered. This even applies to the effect of variations in the pressure downstream of a shock wave on its location, even though the shock wave may not extend completely across the channel. The complete analogy breaks down when flows with large shear are considered in that the Mach number at which bifurcation in solutions is found is not unity, although it is for the case $\delta = O(\epsilon)$.

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